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Kumaraswamy Marshall-Olkin Lindley Distribution: Properties and Applications

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Abstract: In this paper, a new four-parameter distribution is introduced. Moments, conditional moments and moment generating function of the new distribution including are presented. Estimation of its parameters are studied. Two real data applications are described to show its superior performance versus some known lifetime models.

Keywords: Lindley distribution, Marshal-Olkin distribution, Maximum likelihood estimation, Kumaraswamy distribution.

1. Introduction

The Lindley distribution is important for modeling data in biology, medicine and engineering. In recent years, there have been many studies to obtain a new distribution based on medications of the Lindley distribution for fitting such kinds of data. One of the most popular modifications are Marshal-Olkin distribution, Kumaraswamy distribution quadratic rank transmutation map. The main idea of this mapping is to get more flexible structures than the base distribution. Ghitany et al. (2008) studied the properties of the Lindley distribution under a carefully mathematical treatment. They also showed in a numerical example that the Lindley distribution gives better modeling for waiting times and survival times data than the exponential distribution. The use of the Lindley distribution could be a good alternative to analyze lifetime data within the competing risks approach as compared with the use of standard Exponential or even the Weibull distribution commonly used in this area. Many researchers have studied about Lindley distributions. Merovci (2013) proposed a transmuted lindley distribution and applied it to bladder cancer data. Merovci and Elbatal (2014) introduced a transmuted Lindley- geometric distribution and discussed its various properties. They gave an application on real data set which represents the waiting times (in minutes) before service of 100 bank customers to show that the transmuted Lindley-geometric distribution can be a better model than one based on the Lindley geometric distribution and Lindley distribution. Elbatal and Elgarhy (2013) obtained. transmuted quasi Lindley distribution and discussed the least squares, weighted least squares and the maximum likelihood estimation of the parameters of this distribution. Mansour and Mohamed (2015) have introduced a generalization of transmuted Lindley distribution based on a new family of life time distribution and showed that this distribution provided a better model for bladder cancer data than the amongst distributions such as Transmuted Lindley, Exponentiated Lindley, Lindley, Weighted Lindley and Modified Weibull. There are also other studies about different types of Lindley distributions considered by many researchers such as Aryal and Tsokos (2011) and Abdul-Moniem and Seham (2015). Alizadeh et al. (2013) introduced a new family of continuous distributions called the Kumaraswamy Marshal-Olkin generalized family of distributions. They proposed a new extension of the MO family for a given baseline distribution with cdf $(x;\zeta)$, survival function $\bar{G}(x;\zeta) = 1 - G(x;\zeta)$ and pdf $g(x;\zeta)$ depending on a parameter vector ζ. the cdf of the new Kumaraswamy Marshal-Olkin ("KwMO") family of distributions by

$$\mathrm{F(x;a,b,p,\zeta)} = 1 - \left[1 - \left(\frac{G(x;\zeta)}{1 - p\bar{G}(x;\zeta)}\right)^{\mathrm{a}}\right]^{\mathrm{b}} \qquad (1)$$

For $x > 0,\ a > 0,\ b > 0\ and\ \bar{p} > 0.$

The corresponding probability density function is,

$$f(x;\alpha,\beta,\theta,\zeta) = \frac{ab(1-p)g(x;\zeta)G(x;\zeta)^{a-1}}{[1 - p\bar{G}(x;\zeta)]^{\alpha+1}} \times \left(1 - \left[\frac{G(x;\zeta)}{1 - p\bar{G}(x;\zeta)}\right]^{\alpha}\right)^{b-1}. \qquad (2)$$

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The rest of the article is organized as follows. In Section2, introduces the new four-parameter distribution according to Kumaraswamy Marshal-Olkin G-family. In section 3, The Expansion for the pdf and the cdf Functions is derived. Moments, conditional moments and moment generating function of the new distribution including are presented in Section 4. In section 5, we introduce the method of likelihood estimation as point estimation Finally, we fit the distribution to real data set to examine it.

2. A NEW FOUR-PARAMETER DISTRIBUTION

The Lindley distribution was introduced by Lindley (1958) . A random variable X is said to have the Lindley distribution with parameter θ if its probability density is defined as

$$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\qquad x>0,\theta>0. \qquad (3)$$

The corresponding cumulative distribution function (cdf) is:

$$F(x;a,b,p,\theta)=$$

$$1-\left[1-\left(\frac{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}}{1-p\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)}\right)^{a}\right]^{b}, \qquad (5)$$

Hence, the pdf of Kumaraswamy Marshal-Olkin Lindley distribution

$$f(x;a,b,p,\theta)=\frac{ab(1-p)\theta^2(1+x)e^{-\theta x}\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{\alpha-1}}{(\theta+1)\left[1-p\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right]^{\alpha+1}}$$

$$\times\left(1-\left[\frac{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}}{1-p\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)}\right]^{\alpha}\right)^{b-1}. \qquad (6)$$

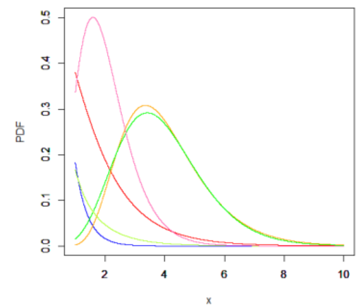
Where for $x>0, a, b, \theta\ and\ \bar{p}>0$ we shall refer to the distribution given by (5) and (6) as the Kumaraswamy Marshall-Olkin Lindley (*KWMOL*).

The failure rate function associated with (6) is given by

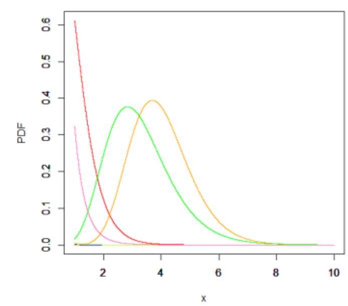
$$h(x;a,b,p,\theta)=\frac{ab(1-p)\,\theta^2(1+x)e^{-\theta x}\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{\alpha-1}}{(\theta+1)\left[1-p\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right]^{\alpha+1}}$$

$$\times\left(1-\left[\frac{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}}{1-p\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)}\right]^{\alpha}\right)^{-1}. \qquad (7)$$

Figure 1 (a) and (b) provide some plots of the *KWMOL* density curves for different values of the parameters a, b, θ and p.



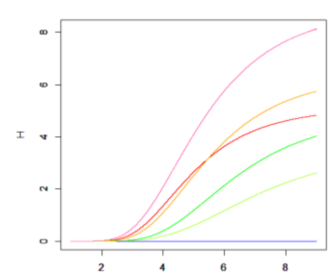
(a)



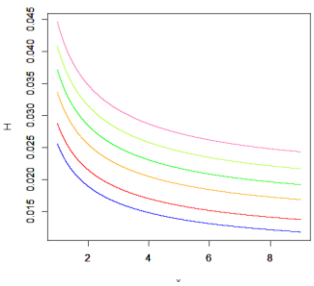
(b)

Figure 1. Plots of the KWMOL density function for some parameter values.(a) For $a=31,, b=2, \theta=3, p=0.3$ (blue line), $a=5, b=3, \theta=0.8, p=0.4$ (red line), $a=11, b=.8, \theta=0.11, p=0.12$ (orange line) $a=9, b=2, \theta=0.9, p=0.8$ (green line) $a=1.5, b=4, \theta=1.5, p=0.3$ (green yellow line), $a=11, b=3, \theta=.9, p=0.6$ (hot pink line) (b) For *a*=1,*b*=4,*θ*=.7 and p=0.5 (blue line), *a*=11, *b*=2 , *θ*=0.6 and p=0.8 (red line),*a*=9, *b*=,7 , *θ* =0.11 and p=0.12 (orange line) *a*=7, *b*=2 , *θ*=0.9 and p=0.9 (green line),*a*=6, *b*=3 , *θ*=1.6 and p=0.7 (green yellow line), *a*=3, *b*=.7 ,*θ*=3 and p=0.8 (hot pink line).

Figure 2 does the same for the associated hazard rate function.



(a)



(b)

Figure 2. Plots of the kwmol hazard rate function for some parameter values. .(a) For a =0.9, b =4, θ =2 and p=.6 (blue line), a=2, , b=4, θ=.7and p=.5 (red line), a=20, b=3, θ=.7 and p=0.6 (orange line) a=9 b=2, θ=0.8and p=0.4 (green line), a=0.7, b=1.5, θ=2.2 and p=0.9 (green yellow line),a=4, b=3, θ=.8and p=0.9 (hot pink line) (b) For a=0.8, b=3, θ=1 and p=0.2 (blue line), a=1, b=2, θ=0.5 and p=0.4 (red line), a=19, b=2, θ=0.8 and p=0.6 (orange line) a=5, b=3, θ=0.5and p=0.3 (green line), a=0.4,b=1, θ=1.2 and p=0.5 (green yellow line),a=5, b=3, θ=0.7 and p=0.6 (hot pink line).

3. EXPANSION FOR THE PDF FUNCTION

In this section we give another expression for the pdf function using the Maclaurin and Binomial expansions for simplifying the pdf cdf form.

Using the expansions

$$(1-z)^{-b} = \sum_{i=0}^{\infty}\binom{-b}{i}(-z)^i, \quad |z|<1, \tag{8}$$

$$e^{-x} = \sum_{i=0}^{\infty}\frac{(-x)^i}{i!}, \tag{9}$$

and

$$(a+b)^k = \sum_{i=0}^{k}\binom{k}{i} b^j a^{k-j}. \tag{10}$$

Using (8),(9) and (10) we can write (6) as

$$f(x;a,b,\theta.p) = \sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} A_{i:m}\, x^m e^{-\theta x(j+k+1)}. \tag{11}$$

Where $A_{i:m}$is a constant term given by:

$$A_{i:m} = \frac{(-1)^{i+j}(l+1)(j+k)!\,\Gamma(b)\Gamma(ai+a+2)a.b.(1-p)\theta^{2+l}p^k}{m!\,(l-m+1)!\,i!\ \ j!\,k!\ \ (j+k_l)!\,(ai+a)\Gamma(ai+a-j)\ \ (\theta+1)^{j+1}}.$$

4. STATISTICAL PROPERTIES

In this section, we derive moments, conditional moments, Moment Generating Function of the *kwmol* distribution.

4.1 Moments

The rth non-central moments or (moments about the origin) of the *kwmol* under using equation (11) is given by:

$$\mu_r' = \int_0^{\infty} x^r \left[\sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} A_{i:m}\, x^m e^{-\theta x(j+k+1)}\right] dx, \tag{13}$$

then

$$\mu_r' = \sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} \boldsymbol{A_{i:m}} \frac{\Gamma(m+r+1)}{[\theta(j+k+1)]^{m+r+}}. \tag{14}$$

4.2 Conditional moments

For lifetime models, it is useful to know the conditional moments defined as $E(x^r|x>t)$,

$$E(x^r|x>t) = \frac{1}{[1-G(t)]}\int_t^{\infty} x^r f(x)dx \tag{15}$$

using equation (11) the conditional moments is,

$$E(x^r|x>t) = \frac{1}{[1-G(t)]}\left[\sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} \boldsymbol{A_{i:m}} \frac{\Gamma_t(m+r+1)}{[\theta(j+k+1)]^{m+r+}}\right], \tag{16}$$

where $\Gamma_t(a) = \int_t^{\infty} x^{a-1}e^{-x}dx$is the upper incomplete gamma function.

4.3 The moment generating function

The moment generating function, $M_x(t)$, can be easily obtained as:

$$M_x(\mathrm{t}) = \int_0^{\infty} e^{tx} f(x) dx, \tag{17}$$

$$M_x(t) = \sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} \boldsymbol{A_{i:m}} \int_0^{\infty} x^m e^{-\theta x(j+k+1)+tx} dx, \tag{18}$$

$$M_x(t) = \int_0^{\infty} x^m\, e^{-x(\theta(j+k+1)+t)}\, dx \tag{19}$$

then, the moment generating function of the *kwmol* distribution is given by,

$$M_x(t) = \sum_{i,j,k=0}^{\infty}\sum_{l=0}^{j+k}\sum_{m=0}^{l+1} A_{i:m} \frac{\Gamma(m+1)}{[\theta(j+k+1)+t]^{m+1}}. \tag{20}$$

5. Estimation of the Parameters

In this section we introduce the method of likelihood to estimate the parameters involved. The maximum likelihood estimators (MLEs) for the parameters of Kumaraswamy Marshal-Olkin Lindley $KWMOL(a,b,\theta,p)$ is discussed in this section. Consider the random sample $x_1, x_2, \ldots, x_n$ of size n from $KWMOL(a,b,\theta,p)$ with probability density function in (6), then the likelihood function can be expressed as follows

$$l=\frac{a^n b^n (1-p)^n \theta^{2n}\prod_{i=1}^{n}(1+x_i)\, e^{-\theta\sum_{i=1}^{n}x_i}}{(\theta+1)^n\prod_{i=1}^{n}\left[1-p\left(\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\right)\right]^{a+1}}$$
$$\times\prod_{i=1}^{n}\left(1-\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\right)^{a-1}$$
$$\times\prod_{i=1}^{n}\left(1-\left[\frac{1-\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}}{1-p\left(\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\right)}\right]^{a}\right)^{b-1}. \quad (21)$$

Where

$$\Theta_i=\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}$$

Hence, the log-likelihood function, $\mathcal{L}$, becomes:

$$\mathcal{L}=n\, ln a+n\ln b+n\, ln(1-p)+2n\ln\theta\sum_{i=1}^{n}ln\ (1+x_i)$$
$$-\theta\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}ln\,(1-\Theta_i)-n\, ln(\theta+1)$$
$$-(a+1)\,ln(1-p\Theta_i)+(b-1)\sum_{i=1}^{n}ln(1-[\frac{1-\Theta_i}{1-p\Theta_i}]^a) \quad (22)$$

Therefore, the MLEs of a, b, θ *and* p must satisfy the following equations:

$$\frac{\partial\mathcal{L}}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}ln\,(1-\Theta_i)-ln(1-p\Theta_i)$$
$$-(b-1)\sum_{i=1}^{n}\frac{(\frac{1-\Theta_i}{1-p\Theta_i})^a ln(\frac{1-\Theta_i}{1-p\Theta_i})}{(1-(\frac{1-\Theta_i}{1-p\Theta_i})^a)}, \quad (23)$$

$$\frac{\partial\mathcal{L}}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}ln\,(1-\left[\frac{1-\Theta_i}{1-p\Theta_i}\right]^a), \quad (24)$$

$$\frac{\partial\mathcal{L}}{\partial p}=\frac{-n}{(1-p)}+(a+1)\frac{\Theta_i}{1-p\Theta_i}$$
$$+(b-1)\sum_{i=1}^{n}\frac{a[\frac{1-\Theta_i}{1-p\Theta_i}]^{a-1}(\frac{(1-\Theta_i)\Theta_i}{(1-p\Theta_i)^2})}{1-(\frac{1-\Theta_i}{1-p\Theta_i})^a}, \quad (25)$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta}=\frac{2n}{\theta}-\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}\frac{\frac{\partial\Theta_i}{\partial\theta}}{(1-\Theta_i)}-\frac{n}{\theta+1}$$
$$+(a+1)\frac{p\frac{\partial\Theta_i}{\partial\theta}}{(1-p\Theta_i)}$$
$$+(b-1)\sum_{i=1}^{n}\frac{(1-p\Theta_i)\frac{\partial\Theta_i}{\partial\theta}+(1-\Theta_i)p\frac{\partial\Theta_i}{\partial\theta}}{[1-(\frac{1-\Theta_i}{1-p\Theta_i})^a](1-p\Theta_i)^2}. \quad (26)$$

The maximum likelihood estimator $\underline{\hat{\vartheta}}=(\hat{a},\hat{\theta},\hat{b},\hat{p})$ of $\underline{\vartheta}=(a,\theta,b,p)$ is obtained by solving the nonlinear system of equations (23) through (26). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

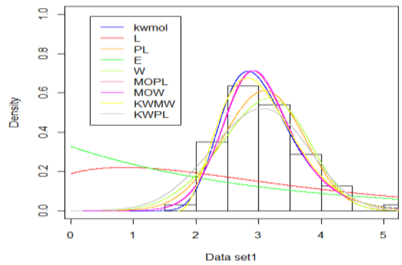
6. Application

In this section, we use two real data sets to show that the Kumaraswamy Marshal-Olkin Lindley (*KWMOL*) distribution can be a better model than nested and non-nested models.

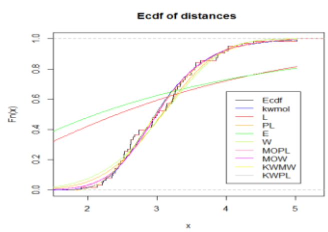
A. Data Set 1

The following data reported by Badar and Priest (1982), which represents the strength measured in GPa for single carbon fibers and impregnated at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 100 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. Here, we consider that the data set of single fibers of 20 mm in gauge with a sample of size 63. The data are: 1.901,2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575,2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030,3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493,3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

In order to compare the two distribution models, we consider criteria like $-2\hat{\ell}$ **,AIC, CAIC, HQIC, BIC,** W^* **and** A^*The better distribution corresponds to smaller KS,$-2\mathcal{L}$, AIC and AICC values.. These numerical results are obtained using R.



(a)



(b)

Figure 3. (a) Estimated densities of the KWMOL, L, PL, E,W, MOPL , MOW, KWMW and KWPL distributions of data set 1.(b) Estimated cdf function from the fitted the KWMOL, L, PL, ,E, W , KWMW,KWPL, MOPL, MOW and the empirical cdf for the data set 1.

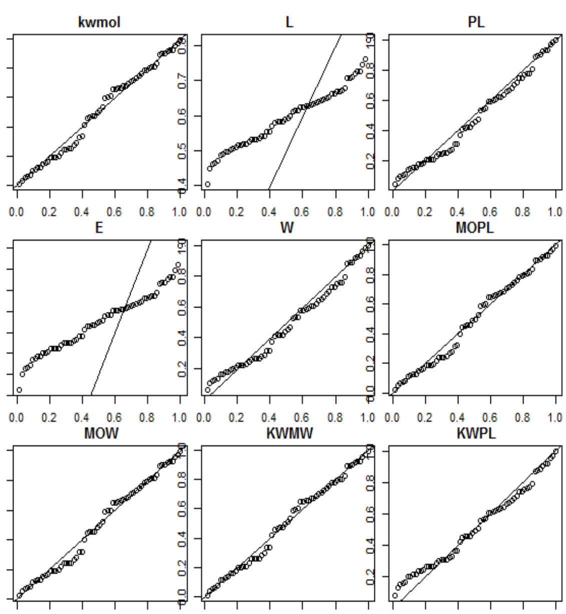
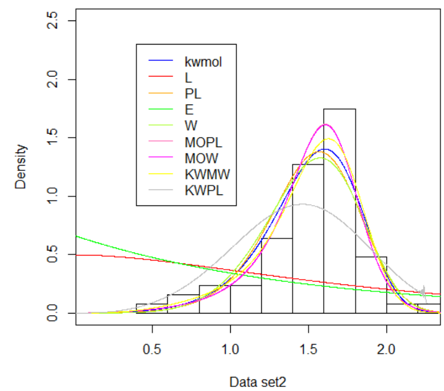


Figure 4. Probability plots for the fits KWMOL, L, PL, E, W, MOPL , MOW, KWMW and KWPL distributions of data set 1.

B. Data Set 2

The data set is obtained from Merovci (2013). The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. Some summary statistics for the data are as follows:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.550	1.375	1.590	1.507	1.685	2.240



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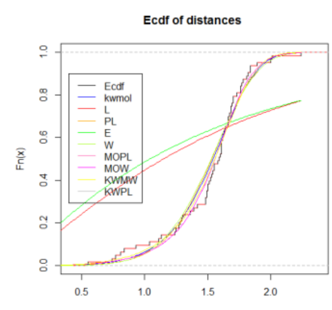

(b)

Figure 5. (a) Estimated densities of the kwmol, L, PL, E,W, MOW , MOPL, KWMW and KWPL distributions of data set 1. for the data set 2. (b) Estimated cdf function from the fitted the kwmol, L, PL, E,W, MOPL , MOW, KWMW, KWPL and the empirical cdf for the data set 2.

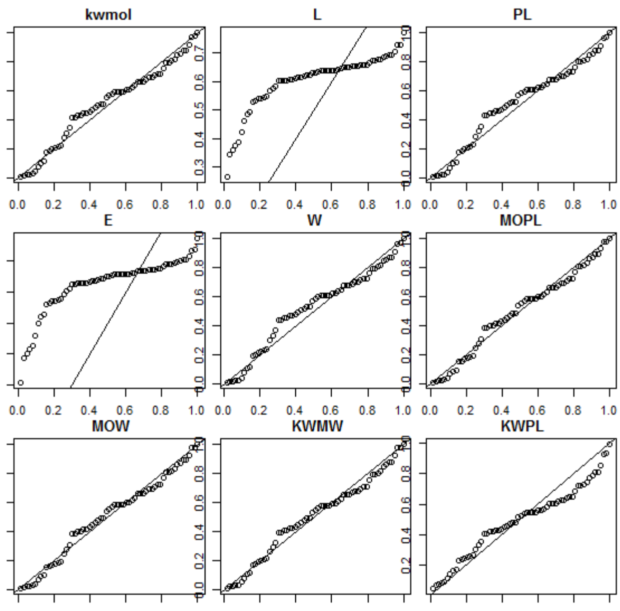

Figure 6. Probability plots for the fits kwmol, L, PL, E,W, MOW , KWMW, MOPLKWPL distributions of data set 2.

As we can see from Tables (1) and (3), our model with smallest values of AIC, AICC, BIC, HQIC$W^*, A^*$and K-S test statstic best fits the data. Figures (3) and (5) shows the empirical distribution compared to the rival models and the fitted densities against the data. We hope that the proposed distribution will serve as an alternative model to other models available in the literature.

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TABLE I. THE STATISTICS $-2\hat{\ell}$, AIC, CAIC, HQIC, BIC, W^* AND A^* FOR THE STRENGTH DATA.

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p-value
KMOL	56.423	120.846	121.53	129.418	124.217	0.064	0.344	,0.082	0.791
L	121.357	244.7	244.08	246.858	245.558	0.060	0.377	,0.430	1.394
PL	59.860	123.7	123.92	128.006	125.406	0.096	0.673	0.089	0.691
E	133.445	268.8	268.95	271.034	269.734	0.058	0.362	0.485	2.408
W	61.956	127.9	128.11	132.200	129.599	0.128	0.892	0.087	0.719
MOPL	99.540	193	192.67	186.651	190.552	0.9876	0.235	0.242	0.02
MOW	57.499	120.9	121.4	127.429	123.528	0.077	0.451	0.088	0.708
KWMW	55.992	121.985	123.03	132.700	126.199	0.0495	0.274	0.074	0.879
KWPL	61.654	131.309	131.99	139.881	134.680	0.099	0.697	0.120	0.322

TABLE II. MLES AND THEIR CORRESPONDING STANDARD ERRORS (IN PARENTHESES) FOR THE STRENGTH DATA.

Model	*Estimates*	
KWMOL	$\hat{a}$ = 19.937428 (30.430) $\hat{b}$ = 1.269832 (0.9)	$\hat{\theta}$ =1.960756 (0.732) $\hat{p}$ = 4.076548(11.92)
L	$\hat{\theta}$ = 0.5392642(0.049)	
PL	$\hat{\beta}$ = 3.62766(0.297)	$\hat{\theta}$ = 0.02825 (0.0107)
E	$\hat{\lambda}$ = 0.32673(0.0411)	
W	$\hat{\lambda}$ = 5.0505 (0.455)	$\hat{\alpha}$ =0.3016883(0.0079)
MOPL	$\hat{p}$ = 3.641(0.007) $\hat{\theta}$ = 0.303 (0.00003)	$\hat{\beta}$ = 0.882(0.00038)
MOW	$\hat{\lambda}$ = 8.3546709(0.969) $\hat{p}$ = 0.9733707(0.0379)	$\hat{\alpha}$ = 0.2157 (0.0302)
KWMW	$\hat{a}$ = 2.991(2.571) $\hat{b}$ = 1.634(0.656) $\hat{\theta}$ = 1.2238(0.328)	$\hat{\gamma}$ =2.733(1.58) $\hat{\beta}$ = 3.085(.732)
KWPL	$\hat{\beta}$ = 3.86 (0.0026) $\hat{b}$ = 0.173 (0.024)	$\hat{\theta}$ =0.077 (0.0017) $\hat{a}$ =0.7299 (0.227)

TABLE III. THE STATISTICS $-2\hat{\ell}$, AIC, CAIC, HQIC, BIC, W^* AND A^* FOR THE STRENGTHS DATA.

Model	$-2\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	W^*	A^*	*KS*	*p-value*
KWMOL	13.863	35.727	36.416	44.299	39.098	0.165	0.920	0.131	0.225
L	81.278	164.55	164.622	166.7	165.399	0.542	2.976	0.386	0.013
PL	14.689	33.379	33.579	37.666	35.0657	0.214	1.179	0.144	0.145
E	88.830	179.660	179.726	181.803	180.503	0.570	3.127	0.417	0.056
W	15.206	34.413	34.613	38.699	36.099	0.237	1.303	0.152	0.108
MOPL	15.031	39.062	40.469	46.491	42.591	0.102	0.576	0.099	0.564
MOW	19.033	38.067	38.474	36.496	32.595	0.105	0.591	0.1	0.554
KWMW	12.732	35.464	36.517	46.180	39.679	0.117	0.671	0.110	0.429
KWPL	21.681	51.362	52.051	59.934	54.733	0.333	1.827	0.246	0.0009

TABLE IV. MLES AND THEIR CORRESPONDING STANDARD ERRORS (IN PARENTHESES) FOR THE BLADDER CANCER DATA.

Model	*Estimates*	
KWMOL	$\hat{a}$ = 2.193(0.879) $\hat{p}$ =24.548(0.987)	$\hat{b}$ = 25.556(0.124) $\hat{\theta}$ = 79.643(0.983)
L	$\hat{\theta}$ = 0.99607(0.094)	
PL	$\hat{\beta}$ =0.222 (0.0466)	$\hat{\theta}$ = 4.454(0.387)
E	$\hat{\lambda}$ = 0.663 (0.083)	
W	$\hat{\alpha}$ = 5.780 (0.576)	$\hat{\lambda}$ = 0.6143(0.0139)
MOPL	$\hat{\theta}$ = 1.090 (0.686) $\hat{p}$ =14.826 (20.59)	$\hat{\beta}$ =2.857(0.748)
MOW	$\hat{\lambda}$ = 3.202 (0.950) $\hat{\alpha}$ =0.892 (0.196)	$\hat{p}$ =15.628 (20.792)
KWMW	$\hat{\theta}$ =0.743 (1.169) $\hat{b}$ = 1.15 (2.87) $\hat{\Gamma}$ =0.0434 (0.0374)	$\hat{a}$ = 4.593(4.1300) $\hat{\beta}$ =6.2166(2.580)
KWPL	$\hat{b}$ =0.230 (0.0762) $\hat{\beta}$ = 5.7593 (0.0025)	$\hat{a}$ = (0.0191) 0.1432 $\hat{\theta}$ =0.368 (0.0025)